Supersymmetric Electromagnetic Waves on Giant Gravitons

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Introduction

- Supersymmetric states have played a crucial role in the development of string theories $\cdots$
  $\cdots$ by helping to uncover and substantiate important aspects about dualities.
- This is especially true of the strong-weak dualities.
- In the context of AdS/CFT the program of matching the BPS spectra on both sides of the duality is still not completed.
- $\cdots$ will be useful in the verification of AdS/CFT in its BPS sectors.
- $\cdots$ may help in accounting for the entropy of the extremal black holes in $AdS_5 \times S^5$.

- In the context of AdS/CFT a significant class of BPS states consists of the so-called giant graviton states of type IIB string theory on $AdS_5 \times S^5$.
- These are supposed to be dual to the R-charged BPS states of the $\mathcal{N} = 4$ $SU(N)$ SYM on S^3 under the AdS/CFT dictionary.
- A lot of progress has been made in counting the R-charged states on the CFT side.
- For instance it is known that the degeneracy of the 1/8-BPS states of $\mathcal{N} = 4$ U(N) SYM is identical to that of N particles in 3 bosonic and 2 fermionic oscillators. [Kinney, Maldacena, Minwalla, Raju]

- A part of this answer has been recovered from the bulk side. [Biswas, Gaiotto, Lahiri, Minwala; Mandal-NVS]
- $\cdots$ this is achieved by starting with classical BPS solutions of D3-branes in $AdS_5 \times S^5$ and quantizing their solution spaces.
- However, we do not yet know the *full* set of BPS solutions of D3-branes with in a fixed susy sector.
- Thus it is important to improve our knowledge of classical solutions of D3-branes in $AdS_5 \times S^5$.

- States in $AdS_5 \times S^5$ are labeled by the eigen values of the commuting set of isometries $(SO(2,4) \times SO(6))$

$$(E, S_1, S_2;\ J_1, J_2, J_3)$$

- BPS states preserve at least 2 susy out of the 32 of $AdS_5 \times S^5$, and satisfy

$$E = |S_1| + |S_2| + |J_1| + |J_2| + |J_3|.$$

- BPS D3-brane configurations in $AdS_5 \times S^5$ fall into two categories:

Giant

A D3-brane on a 3-surface in S^5, point-like in AdS_5 that moves along a time-like geodesic.

Dual-Giant

A D3-brane on a 3-surface in AdS_5, point-like in S^5 and moving along a maximal circle.

- Several examples are known:
  characterized by their susy and non-zero charges.

- There are three sectors with (at least) 4 susy:

$$(J_1, J_2, J_3), \quad (S_1, J_1, J_2,), \quad (S_1, S_2, J_1).$$

- Part of the (J_1, J_2, J_3) sector has two descriptions:
  (i) Mikhailov giants [Mikhailov] (ii) spinning dual-giants [Mandal-NVS]
- Part of the (S_1, S_2, J_1) sector also has two descriptions
  (i) Spinning giants [Mandal-NVS] (ii) Wobbling dual-giants [Ashok-NVS]
- In all these solutions only the transverse scalars of D3-branes excited.
- Mikailov giants and Wobbling dual-giants are the full set of solutions with the scalars turned on in their susy sectors. [Ashok-NVS]

- In general one expects solutions with transverse scalars, the gauge fields and fermions turned on.
- Some isolated examples of BPS gauge field configurations on round S^3 giants (1/2-BPS) were known [Kim, Lee; Sinha, Sonner]
- In this talk we address the problem of finding *all* giant and dual-giant like solutions of a D3-brane in $AdS_5 \times S^5$ with scalars and electromagnetic fields on its world-volume turned on.

Result

- Our solutions can be elegantly described using the auxiliary space $\mathbb{C}^{1,2} \times \mathbb{C}^3$ with coordinates $\{\Phi^0, \Phi^1, \Phi^2; Z_1, Z_2, Z_3\}$ where the $AdS_5 \times S^5$ can be embedded as

$$|\Phi^0|^2 - |\Phi^1|^2 - |\Phi^2|^2 = l^2, \quad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = l^2.$$

Turning on gauge field does not change the embedding geometry of the D3-brane.

- So the D3-brane embeddings continue to be given by the Wobbling dual-giants and Mikhailove giants.

Wobbling dual-giants:

Defining $Y^i = \Phi^i Z_1$, these are described as the intersection of

$$\begin{aligned} f(Y^0, Y^1, Y^2) = 0 \quad &\text{and} \\ Z_2 = Z_3 = 0 & \end{aligned} \tag{1}$$

with $AdS_5 \times S^5$.

Mikhailov giants:

These are the intersections of $AdS_5 \times S^5$ with

$$\begin{aligned} f(X^1, X^2, X^3) = 0 \quad &\text{and} \\ \Phi^1 = \Phi^2 = 0\,, & \end{aligned} \tag{2}$$

where $X^i = Z_i \Phi^0$.

- Assuming that F is given by the pull-back of a bulk 2-form we find

For dual-giants

The susy world-volume gauge field F is given by the real part of

$$G = \sum_{ij=0,1,2} G_{ij}(Y) dY^i \wedge dY^j .$$

pulled back onto the world-volume.

For giants

The BPS world-volume gauge field strength is again given by the real part of

$$\tilde{G} = \sum_{ij=1,2,3} \tilde{G}_{ij}(X)\, dX^i \wedge dX^j , \tag{3}$$

pulled back onto the world-volume.

Computations

- To find supersymmetric configurations of a D-brane with bosonic fields it is sufficient to impose the κ-projection condition.
- We will solve the kappa-projection conditions a D3-brane embedded in $AdS_5 \times S^5$ with world-volume gauge field flux F.
- The metric on $AdS_5 \times S^5$ written in global coordinates is

$$\frac{ds^2}{l^2} = -\cosh^2\rho\, d\phi_0^2 + d\rho^2 + \sinh^2\rho\,(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2) + d\alpha^2 + \sin^2\alpha\, d\xi_1^2 + \cos^2\alpha\,(d\beta^2 + \sin^2\beta\, d\xi_2^2 + \cos^2\beta\, d\xi_3^2)$$

where $\phi_0 = \frac{t}{l}$.

- This corresponds to parametrizing AdS_5 in $\mathbb{C}^{1,2}$ as

$$\begin{aligned}\Phi^0 &= l\cosh\rho\, e^{i\phi_0}\,,\\ \Phi^1 &= l\sinh\rho\cos\theta\, e^{i\phi_1}\,,\\ \Phi^2 &= l\sinh\rho\sin\theta\, e^{i\phi_2}\,.\end{aligned} \tag{4}$$

- And S^5 in $\mathbb{C}^3$ is parametrized as

$$\begin{aligned}Z_1 &= l\sin\alpha\, e^{i\xi_1}\\ Z_2 &= l\cos\alpha\,\sin\beta\, e^{i\xi_2}\\ Z_3 &= l\cos\alpha\,\cos\beta\, e^{i\xi_3}\end{aligned} \tag{5}$$

- SUSY analysis needs an orthonormal frame for the $AdS_5 \times S^5$ metric $\cdots$

- We choose the following frame for the AdS_5 part of the metric

$$
\begin{aligned}
e^0 &= l[\cosh^2\rho\, d\phi_0 - \sinh^2\rho\,(\cos^2\theta d\phi_1 + \sin^2\theta d\phi_2)],\\
e^1 &= l\, d\rho\,, \qquad e^2 = l \sinh\rho\, d\theta,\\
e^3 &= l\cosh\rho \sinh\rho\,(\cos^2\theta\; d\phi_{01} + \sin^2\theta\; d\phi_{02})\\
e^4 &= l \sinh\rho\, \cos\theta \sin\theta\; d\phi_{12}
\end{aligned}
\tag{6}
$$

  where $\phi_{ij} = \phi_i - \phi_j$.
- The base is the Kähler manifold $\widetilde{\mathbb{CP}}^2$ along $\{r, \theta, \phi_{01}, \phi_{02}\}$, and the fibre along $\phi_0 + \phi_1 + \phi_2$.
- The Kähler form on $\widetilde{\mathbb{CP}}^2$ is $\tilde{\omega} = e^{13} + e^{24}$.

- For the S^5 part we choose the frame

$$
\begin{aligned}
e^5 &= l\, d\alpha, \qquad e^6 = l \cos\alpha\, d\beta, \\
e^7 &= l \cos\alpha \sin\alpha\, (\sin^2\beta\, d\xi_{12} + \cos^2\beta\, d\xi_{13}), \\
e^8 &= l \cos\alpha \cos\beta \sin\beta\, d\xi_{23}, \\
e^9 &= l\, (\sin^2\alpha\, d\xi_1 + \cos^2\alpha \sin^2\beta\, d\xi_2 + \cos^2\alpha \cos^2\beta\, d\xi_3)
\end{aligned}
\tag{7}
$$

  where $\xi_{ij} = \xi_i - \xi_j$.
- This makes manifest the fact that S^5 is a Hopf fibration.
- The base is the Kähler manifold $\mathbb{CP}^2$ along $\{\alpha, \beta, \xi_{12}, \xi_{13}\}$ coordinates, and the fibre is along $\xi_1 + \xi_2 + \xi_3$.
- The Kähler form on $\mathbb{CP}^2$ is $\omega = e^{57} + e^{68}$.

- $AdS_5 \times S^5$ is a maximally susy solution of type IIB.
- The Killing spinor adapted to the above frame is

$$\epsilon = e^{-\frac{1}{2}(\Gamma_{79}-i\Gamma_5\,\tilde{\gamma})\,\alpha} e^{-\frac{1}{2}(\Gamma_{89}-i\Gamma_6\tilde{\gamma})\beta}\, e^{\frac{1}{2}\xi_1\Gamma_{57}}\, e^{\frac{1}{2}\xi_2\Gamma_{68}}\, e^{\frac{i}{2}\xi_3\Gamma_9\,\tilde{\gamma}} \\ \times\, e^{\frac{1}{2}\rho\,(\Gamma_{03}+i\Gamma_1\,\gamma)}\, e^{\frac{1}{2}\theta\,(\Gamma_{12}+\Gamma_{34})}\, e^{\frac{i}{2}\phi_0\,\Gamma_0\,\gamma}\, e^{-\frac{1}{2}\phi_1\Gamma_{13}}\, e^{-\frac{1}{2}\phi_2\Gamma_{24}}\,\epsilon_0$$

- ϵ_0 is an arbitrary 32-component constant weyl spinor

$$\Gamma_0\cdots\Gamma_9\epsilon_0 = -\epsilon_0,\ \gamma = \Gamma^{01234},\ \tilde{\gamma} = \Gamma^{56789}.$$

- We seek the full set of BPS equations for a single D3-brane preserving two susy out of the 32.
- We could choose the projections (not unique)

$$\begin{aligned} \Gamma_{57}\epsilon_0 = \Gamma_{68}\epsilon_0 &= i\epsilon_0\,, \\ \Gamma_{09}\epsilon_0 &= -\epsilon_0\,, \\ \Gamma_{13}\epsilon_0 = \Gamma_{24}\epsilon_0 &= -i\epsilon_0. \end{aligned}$$

- With these projections the killing spinor simplifies to

$$\epsilon = e^{\frac{i}{2}(\phi_0+\phi_1+\phi_2+\xi_1+\xi_2+\xi_3)}\epsilon_0.$$

- We take the most general D3-brane ansatz $\cdots$
- $\cdots$ all the space-time coordinates

$$(t, r, \theta, \phi_1, \phi_2, \alpha, \beta, \xi_1, \xi_2, \xi_3)$$

are functions of all the world-volume coordinates

$$(\sigma^0, \sigma^1, \sigma^2, \sigma^3).$$

- The kappa projection condition is

$$\epsilon^{ijkl}\left[\gamma_{ijkl} I + F_{ij}\gamma_{kl} IK + F_{ij}F_{kl} I\right]\epsilon = \sqrt{-\det(h+F)}\ \ \epsilon$$

[Cederwall, von Gussich, Nilsson, Westerberg; Bergshoeff, Townsend]

- Here, the world-volume gamma matrices are

$$\gamma_i = \mathfrak{e}_i^a\,\Gamma_a, \quad \text{where} \quad \mathfrak{e}_i^a = e_\mu^a\,\partial_i X^\mu,$$

with $i \in \{\tau, \sigma_1, \sigma_2, \sigma_3\}$ and $h_{ij} = \eta_{ab}\mathfrak{e}_i^a\mathfrak{e}_j^b$ is the induced metric.

- The operators K and I act as

$$K\epsilon = \epsilon^* \qquad \text{and} \qquad I\epsilon = -i\epsilon\,.$$

- Procedure:
  - Project the κ-equation on to some $\bar{\chi}_0$ and use the projection conditions to simplify it down to –
  - – a linear combination of non-vanishing and independent spinor bilinears $\bar{\chi}_0 \Gamma_{ab\cdots}\epsilon_0$ and $\bar{\chi}_0 \Gamma_{ab\cdots}\epsilon_0^*$.
  - Finally $\cdots\cdots$ set their coefficients to zero.
- We have chosen to preserve the same susy as for the giant gravitons without electromagnetic flux.
- This means we have the condition

$$\gamma_{\sigma_0\sigma_1\sigma_2\sigma_3}\epsilon = i\sqrt{-\det h}\,\epsilon$$

where $\gamma_{\sigma_0\sigma_1\sigma_2\sigma_3} = \mathfrak{e}_0^a \mathfrak{e}_1^b \mathfrak{e}_2^c \mathfrak{e}_3^d \Gamma_{abcd}$.

- This leads to the constraint

$$\epsilon^{ijkl} F_{ij} \gamma_{kl} \epsilon_0^* = 0\,. \tag{8}$$

- This, in turn, gives rise to the condition

$$\sqrt{-\det h} + \mathrm{Pf}[F] = \sqrt{-\det(h+F)}\,, \tag{9}$$

where $\mathrm{Pf}[F] = \frac{1}{8}\epsilon^{ijkl} F_{ij} F_{kl}$ which we sometimes denote by "$F \wedge F$".

- To write the BPS equations in a compact form, define the complex one-forms

$$\mathbf{E}^1 = \mathfrak{e}^1 - i\mathfrak{e}^3,\ \mathbf{E}^2 = \mathfrak{e}^2 - i\mathfrak{e}^4,\ \mathbf{E}^5 = \mathfrak{e}^5 + i\mathfrak{e}^7,\ \mathbf{E}^6 = \mathfrak{e}^6 + i\mathfrak{e}^8\,,$$

and the real 1-forms

$$\mathbf{E}^{\mathbf{0}} = \mathfrak{e}^0 + \mathfrak{e}^9 \qquad \text{and} \qquad \mathbf{E}^{\bar{0}} = \mathfrak{e}^0 - \mathfrak{e}^9$$

- It can be shown that the equation (8) linear in F gives

$$\begin{aligned} F\wedge \mathbf{E}^{AB} &= 0 \quad \text{for} \quad A,B=\{1,2,5,6\} \\ F\wedge \mathbf{E}^{0}\wedge \mathbf{E}^{A} &= 0 \quad \text{for} \quad A=\{1,2,5,6\} \\ F\wedge (\mathfrak{e}^{09}+i\Omega) &= 0\,, \end{aligned} \tag{10}$$

where $\Omega=\tilde{\omega}-\omega$, with

$$\begin{aligned} \tilde{\omega} &= \mathfrak{e}^{13}+\mathfrak{e}^{24} = -\frac{i}{2}(\mathbf{E}^{1\bar{1}}+\mathbf{E}^{2\bar{2}}) \quad \text{and} \\ \omega &= \mathfrak{e}^{57}+\mathfrak{e}^{68} = \ \frac{i}{2}(\mathbf{E}^{5\bar{5}}+\mathbf{E}^{6\bar{6}})\,. \end{aligned}$$

- In these equations, by $F\wedge \mathbf{E}^{ab}$ we mean $\frac{1}{2}\epsilon^{ijkl}F_{ij}\mathbf{E}^{a}_{k}\mathbf{E}^{b}_{l}$ etc.
- Next we would like to solve equation (9).

- For this we note the following identity

$$
\begin{aligned}
-\det(h+F) = -\det h - (\mathrm{Pf}[F])^2 + (\mathfrak{e}^{09}\wedge F)^2 \\
- \sum_{A=1,2,5,6}\left[|\mathfrak{e}^{9}\wedge \mathbf{E}^{A}\wedge F|^2 - |\mathfrak{e}^{0}\wedge \mathbf{E}^{A}\wedge F|^2\right] \\
- \sum_{A<B}|\mathbf{E}^{AB}\wedge F|^2 - (\Omega\wedge F)^2 + (\Omega\wedge\Omega)\,\mathrm{Pf}[F]\,.
\end{aligned}
$$

- Substituting the BPS conditions linear in the field strength into this expression, and noting that $\Omega\wedge\Omega = \frac{1}{2}(\tilde{\omega}-\omega)\wedge(\tilde{\omega}-\omega) = 0$ for time-like D3-branes (see [Ashok-NVS] for details) we obtain

$$\det(h+F) = \det h + (F\wedge F)^2\,. \tag{11}$$

Demanding the consistency of (9, 11) we get the last of the F-dependent BPS conditions

$$F\wedge F = 0\,. \tag{12}$$

This in turn implies $\det(h+F) = \det h$ for the BPS configurations we seek.

- Finally the BPS eqns that do not involve field strength F are:

$$\mathbf{E}^{ABCD} = 0, \quad \mathbf{E}^{0ABC} = 0$$
$$(\mathfrak{e}^{09} + i\,\Omega) \wedge \mathbf{E}^{AB} = 0 \ \text{ for } \ A, B, C, D = 1, 2, 5, 6$$
$$\Omega \wedge \Omega = 0\,.$$

for time-like brane embeddings.

- In these equations we understand $\mathbf{E}^{abcd}$ to be the function (0-form) $\epsilon^{ijkl}\mathbf{E}_i^a\mathbf{E}_j^b\mathbf{E}_k^c\mathbf{E}_l^d$ etc.
- Using all the BPS conditions, for a time-like D3-brane one obtains

$$\sqrt{-\det h} = e^{09} \wedge \Omega = i\,\mathbf{E}^{0\bar{0}} \wedge \sum_A \mathbf{E}^{A\bar{A}}\,. \qquad (13)$$

- This is the “calibrating" form.

- For dual-giants, the BPS equations take the simplified form

$$\begin{aligned} \mathbf{E}^{0\bar{0}12} &= 0, \\ \mathbf{E}^0 \wedge \{\mathbf{E}^1, \mathbf{E}^2\} \wedge \tilde{\omega} &= 0, \\ \mathbf{E}^5 = \mathbf{E}^6 &= 0\,, \end{aligned} \tag{14}$$

while for giants, they take the form

$$\begin{aligned} \mathbf{E}^{0\bar{0}56} &= 0, \\ \mathbf{E}^0 \wedge \{\mathbf{E}^5, \mathbf{E}^6\} \wedge \omega &= 0, \\ \mathbf{E}^1 = \mathbf{E}^2 &= 0\,. \end{aligned} \tag{15}$$

- We now restrict our attention to dual-giant gravitons.

- Using the fact that the field-strength F is real, the F-dependent BPS conditions for the dual-giants take the form

$$
\begin{aligned}
F \wedge \mathbf{E}^{0\bar{0}} &= 0 \\
F \wedge \mathbf{E}^{0} \wedge \{\mathbf{E}^{1}, \mathbf{E}^{2}, \mathbf{E}^{\bar{1}}, \mathbf{E}^{\bar{2}}\} &= 0 \\
F \wedge F &= 0 \\
F \wedge (\mathbf{E}^{1\bar{1}} + \mathbf{E}^{2\bar{2}}) &= 0 \\
F \wedge \{\mathbf{E}^{12}, \mathbf{E}^{\bar{1}\bar{2}}\} &= 0\,.
\end{aligned}
\tag{16}
$$

- Next, we turn to solving these equations.

- At this point we make an assumption, that the field strength F on the world-volume can be written as a pull-back of a space-time 2-form onto the world-volume.
- This assumption allows us to solve the above algebraic conditions in a rather straightforward way.
- There are fifteen 2-forms that can be constructed out of the six bulk 1-forms of relevance $\{\mathbf{E}^0, \mathbf{E}^{\bar{0}}, \mathbf{E}^1, \mathbf{E}^2, \mathbf{E}^{\bar{1}}, \mathbf{E}^{\bar{2}}\}$ and the 2-form we seek is a real linear combination of all these two-forms.

- We start by assuming the most general ansatz for F:

$$F = \mathrm{Re}\left[\chi_{0\bar{0}}\mathbf{E}^{0\bar{0}} + \sum_A (\chi_{0A}\mathbf{E}^{0A} + \chi_{\bar{0}A}\mathbf{E}^{\bar{0}A}) + \sum_{A\leq B}(\chi_{AB}\mathbf{E}^{AB} + \chi_{A\bar{B}}\mathbf{E}^{A\bar{B}})\right]$$

- After using the BPS equations one will still be left with linear combinations of nine of the non-vanishing 4-forms on the left hand side of.
- We treat these nine 4-forms to be independent and set their coefficients to zero.
- This is justified because of our assumption that F can be written as the pull-back of a space-time 2-form irrespective of the details of the world-volume embedding equations.

- With this assumption it can be shown that the algebraic conditions can be solved if and only if

$$F = \mathrm{Re}[\chi_{01}\mathbf{E}^{01} + \chi_{02}\mathbf{E}^{02} + \chi_{12}\mathbf{E}^{12}] \tag{17}$$

  where $\chi_{01}, \chi_{02}, \chi_{12}$ are arbitrary complex functions of the bulk coordinates restricted to the world-volume.
- It now remains to solve for the parameters $\{\chi_{01}, \chi_{02}, \chi_{12}\}$ by imposing the Bianchi identity and the equation of motion for the gauge field.
- These are

$$dF = 0 \qquad \text{and} \qquad \partial_i X^{ij} = 0\,, \tag{18}$$

  where

$$X^{ij} = \frac{1}{2}\sqrt{-\det(h+F)}\left[(h+F)^{-1} - (h-F)^{-1}\right]^{ij}. \tag{19}$$

- For any 4×4 symmetric matrix h whose components can be written as $h_{ij} = \mathfrak{e}_i^a \mathfrak{e}_j^b \eta_{ab}$ and for any antisymmetric 4×4 matrix F, we note the identity

$$\det(h+F)[(h+F)^{-1} - (h-F)^{-1}]^{ij}$$
$$= -(\frac{1}{4}\epsilon^{pqrs} F_{pq} F_{rs})\, \epsilon^{ijmn} F_{mn} - (\frac{1}{2}\epsilon^{pqrs} F_{pq} \mathfrak{e}_r^a \mathfrak{e}_s^b)\, \eta_{ac}\eta_{bd}\epsilon^{ijmn} \mathfrak{e}_m^c \mathfrak{e}_n^d \,.$$

- Given the definition of X^{ij} in and using the BPS equation $F \wedge F = 0$, we obtain

$$X^{ij} = \frac{1}{2\sqrt{-\det h}}(\frac{1}{2}\epsilon^{pqrs} F_{pq} \mathfrak{e}_r^a \mathfrak{e}_s^b)\, \eta_{ac}\eta_{bd}\epsilon^{ijmn}\, \mathfrak{e}_m^c \mathfrak{e}_n^d \,.$$

- We will need to simplify this further using the BPS equations.
- Before proceeding further we note that the equation of motion $\partial_i X^{ij} = 0$ can be written as $d\tilde{X} = 0$ for the 2-form defined as

$$\tilde{X} = \frac{1}{4}\epsilon_{ijmn} X^{mn}\, d\sigma^i \wedge d\sigma^j .$$

- We will therefore work with $\tilde{X}$ and simplify it using our ansatz for the field strength F and the BPS equations.
- Substituting the ansatz we have for F and retaining only those terms which (potentially) survive after using the BPS equations one finds

$$\tilde{X} = -\frac{1}{\sqrt{-\det h}}\Big[(F \wedge \mathbf{E}^{\bar{0}\bar{1}})\,\mathbf{E}^{01} + (F \wedge \mathbf{E}^{\bar{0}1})\,\mathbf{E}^{0\bar{1}} + (F \wedge \mathbf{E}^{\bar{0}\bar{2}})\,\mathbf{E}^{02} + (F \wedge \mathbf{E}^{\bar{0}2})\,\mathbf{E}^{0\bar{2}}\Big] , \quad (20)$$

where we have re-expressed $\mathfrak{e}^{ab}$ in terms of $\mathbf{E}^{ab}$.

- It can be shown that when $F = \mathrm{Re}[\chi_{01}\,\mathbf{E}^{01} + \chi_{02}\,\mathbf{E}^{02} + \chi_{12}\,\mathbf{E}^{12}]$

$$\tilde{X} = \frac{i}{\sqrt{-\det h}}(\mathbf{E}^{0\bar{0}1\bar{1}} + \mathbf{E}^{0\bar{0}2\bar{2}})\,\mathrm{Im}[\chi_{01}\,\mathbf{E}^{01} + \chi_{02}\,\mathbf{E}^{02} + \chi_{12}\,\mathbf{E}^{12}]\,.$$

- For this we had to use the BPS equations and the identity

$$\mathbf{E}^{a[bcd}\,\mathbf{E}^{f]a} = 0\,,$$

  where, as before, we understand $\mathbf{E}^{abcd}$ to mean $\epsilon^{ijkl}\mathbf{E}_i^a\mathbf{E}_j^b\mathbf{E}_k^c\mathbf{E}_l^d$, and treat $\mathbf{E}^{ab}$ as the rank-2 anti-symmetric tensor $\mathbf{E}_i^a\mathbf{E}_j^b - \mathbf{E}_j^a\mathbf{E}_i^b$.
- Finally restricting to the case of dual-giants, we have

$$\sqrt{-\det h} = i(\mathbf{E}^{0\bar{0}1\bar{1}} + \mathbf{E}^{0\bar{0}2\bar{2}})\,.$$

- Thus we finally obtain the result

$$\tilde{X} = \mathrm{Im}[\chi_{01}\mathbf{E}^{01} + \chi_{02}\mathbf{E}^{02} + \chi_{12}\mathbf{E}^{12}]\,. \qquad (21)$$

- This is a remarkable simplification, considering the original expression we started with, which was highly non-linear in the pull-back of the vielbeins and the field strength F.
- This can be attributed to the effectiveness of the BPS equations in simplifying the problem.
- Our final expressions for the real 2-forms F and $\tilde{X}$ make it natural to define a complex 2-form

$$G = F + i\tilde{X} = \chi_{01}\mathbf{E}^{01} + \chi_{02}\mathbf{E}^{02} + \chi_{12}\mathbf{E}^{12} \tag{22}$$

in terms of which the Bianchi identity and the equations of motion can be combined into the single equation

$$dG = 0\,,$$

where dG refers to the exterior derivative of G on the world-volume.

- However, for differential forms, the pull-back operation and the exterior differentiation commute.
- So, we treat G as a spacetime 2-form and compute the exterior derivative in spacetime, and then require that the resulting 3-form vanishes, when pulled back onto the world-volume.
- Let us first recall some facts regarding the wobbling dual-giant solution. It is known that a wobbling dual-giant is described by a polynomial equation of the form

$$f(Y_0, Y_1, Y_2) = 0\,, \tag{23}$$

where $Y^i = \Phi^i Z_1$ with

$$\begin{aligned}&\Phi^0 = \cosh\rho\, e^{i\phi_0}, \quad \Phi^1 = \sinh\rho\cos\theta\, e^{i\phi_1}, \quad \Phi^2 = \sinh\rho\sin\theta\, e^{i\phi_2}\\ &Z_1 = e^{i\xi_1} \ \text{ and } \ Z_2 = Z_3 = 0\,.\end{aligned} \tag{24}$$

- On such a $3+1$ dimensional world-volume, we seek a closed complex 2-form of the kind (22).
- Since the equation of the D-brane is written purely in terms of the Y^i variables we rewrite (22) in terms of the differentials dY^i.
- Given the definition of the Y^i above, one can relate the differentials dY^i to the 1-forms E^A and $E^{\bar{A}}$. Using these one finds:

$$\begin{aligned} G &= \chi_{01}\mathbf{E}^{01} + \chi_{02}\mathbf{E}^{02} + \chi_{12}\mathbf{E}^{12} \\ &:= G_{01}\frac{dY^0}{Y^0}\wedge\frac{dY^1}{Y^1} + G_{02}\frac{dY^0}{Y^0}\wedge\frac{dY^2}{Y^2} + G_{12}\frac{dY^1}{Y^1}\wedge\frac{dY^2}{Y^2}\,. \end{aligned} \quad (25)$$

- Here the G_{ij} are given in terms of the χ_{ij} which can be inverted to express the χ_{ij} as linear combinations of G_{ij} since the matrix of coefficients is non-singular.

- The relations between the 2-form are

$$
\begin{aligned}
\frac{dY^0}{Y^0}\wedge\frac{dY^1}{Y^1} &= \frac{i}{\sinh\rho}\left[\frac{1}{\cosh\rho}E^{01}-\tan\theta\, E^{02}\right]-\frac{\tan\theta}{\cosh\rho}E^{12},\\
\frac{dY^0}{Y^0}\wedge\frac{dY^2}{Y^2} &= \frac{i}{\sinh\rho}\left[\frac{1}{\cosh\rho}E^{01}+\cot\theta\, E^{02}\right]+\frac{\cot\theta}{\cosh\rho}E^{12},\\
\frac{dY^1}{Y^1}\wedge\frac{dY^2}{Y^2} &= \frac{1}{\sinh\rho\cos\theta\sin\theta}\left[iE^{02}+\coth\rho\, E^{12}\right]. \qquad (26)
\end{aligned}
$$

- These combinations of $\{E^{01}, E^{02}, E^{12}\}$ have the important property that their exterior derivatives vanish - a useful fact.

- Let us now turn to solving the equation $dG = 0$. The left hand side of this equation reads

$$dG = dG_{01} \wedge \left(\frac{i}{\sinh\rho} \left[\frac{1}{\cosh\rho} \mathbf{E}^{01} - \tan\theta\, \mathbf{E}^{02} \right] - \frac{\tan\theta}{\cosh\rho}\, \mathbf{E}^{12} \right) \\ + dG_{02} \wedge \left(\frac{i}{\sinh\rho} \left[\frac{1}{\cosh\rho} \mathbf{E}^{01} + \cot\theta\, \mathbf{E}^{02} \right] + \frac{\cot\theta}{\cosh\rho}\, \mathbf{E}^{12} \right) \\ + dG_{12} \wedge \left(\frac{1}{\sinh\rho\cos\theta\sin\theta} \left[i\mathbf{E}^{02} + \coth\rho\, \mathbf{E}^{12} \right] \right), \quad (27)$$

with

$$dG_{ij} = (K_0 G_{ij})\, \mathbf{E}^0 + (K_1 G_{ij})\, \mathbf{E}^1 + (K_2 G_{ij})\, \mathbf{E}^2 + (K_{\bar{0}} G_{ij})\, \mathbf{E}^{\bar{0}} \\ + (K_{\bar{1}} G_{ij})\, \mathbf{E}^{\bar{1}} + (K_{\bar{2}} G_{ij})\, \mathbf{E}^{\bar{2}},$$

where K_A is the vector field dual to the 1-form E^A.

- Now, (27) is an equation for a 3-form on the world-volume of the dual-giant and one should set the coefficients of the linearly independent 3-forms to zero. As before, we will do this pretending that this is a bulk 3-form equation given the form of our ansatz. Such a solution would lead to a spacetime 2-form G, independent of the particular polynomial that defines the dual-giant. We implement this procedure below.
- Given that the equation describing the dual-giant is holomorphic in the variables Y^i, it follows that $\mathbf{E}^{012} = 0$. This is true irrespective of the precise form of the defining polynomial $f(Y^i) = 0$.
- Also, from (27), it follows that two of the three indices in the 3-form have to be unbarred. Given these constraints, there are precisely nine independent 3-forms that appear on the right hand side of that equation if we substitute into (27).

- After some algebra we find that imposing $dG = 0$ is equivalent to imposing the nine constraints

$$K_{\bar{0}} G_{ij} = K_{\bar{1}} G_{ij} = K_{\bar{2}} G_{ij} = 0 \quad \text{for} \quad i, j \in \{0, 1, 2\}\,. \tag{28}$$

- These make G_{ij} to be functions of Y^0, Y^1, Y^2 and not their conjugates.
- We can summarize our results so far as follows:

- Any 1/8-BPS dual-giant in $AdS_5 \times S^5$ with non-trivial world-volume electromagnetic fields is specified by
  - a holomorphic function
    $$f(Y^0, Y^1, Y^2) \quad \text{and}$$
  - a holomorphic 2-form
    $$G = \sum_{i,j=0,1,2} G_{ij} \frac{dY^i}{Y^i} \wedge \frac{dY^j}{Y^j},$$
    with $G_{ij} = G_{ij}(Y^0, Y^1, Y^2)$.
- The world-volume is obtained by taking the intersection of the zero-set of $f(Y^0, Y^1, Y^2)$ with $AdS_5 \times S^5$ and the field strength of the world-volume gauge field is given by the real part of the 2-form G pulled back onto the world-volume.
- This completes our general analysis of the BPS equations for dual-giants in the $AdS_5 \times S^5$ background of type IIB supergravity.
- Similar analysis gives rise to the corresponding result for giants.

Conclusion

- We have (with an assumption) generalized all the Mikhailov giants and Wobbling dual-giants to include the world-volume gauge field.
- We have a world-volume reparametrization invariant description.
- What about fermions?
- What about charges and quantization?